A major challenge in teaching mathematics is helping students make genuine sense of mathematical ideas. One proven way to meet this challenge is to base teaching on research of students’ mathematical thinking (Bransford, Brown, and Cocking 1999; Fennema et al. 1996). In fact, to produce genuine understanding and powerful mathematical reasoning in students, teaching must be guided by detailed research-based knowledge about the ways that students think about particular mathematical ideas. Such knowledge is critical in selecting and creating instructional tasks, asking appropriate questions, guiding classroom discussions, adapting instruction to students’ needs, understanding students’ reasoning, assessing students’ learning progress, and diagnosing and remediating students’ learning difficulties.

This article describes assessment tasks and a conceptual framework that can be used to understand elementary students’ thinking about the concept of length. The tasks and framework were created in a project in which numerous elementary school students were interviewed to investigate their understanding of various topics in elementary school mathematics. This project—called Cognition Based Assessment (CBA)—has created sets of assessment tasks and research-based conceptual frameworks for describing students’ reasoning about core ideas in elementary school mathematics (Battista 2001, 2004).

The Concept of Length: Critically Important, Deceptively Complicated

The concept of length is critically important in both daily life and formal geometry. In everyday life, we use length to describe how big objects are and how far we travel. In geometry, we use length to add precision to descriptions of shapes. For example, we define a square by saying that it is a quadrilateral with four right angles and four sides of equal length. Without the concept of length, many descriptions of the spatial world become vague and imprecise.

Despite its importance and apparent simplicity, however, the concept of length can be very difficult for children to understand. One reason for this difficulty is that everyday use of the word length is often inconsistent with mathematical use. For instance, the Oxford English Dictionary (OED) defines length as “the linear magnitude of any thing as measured from end to end.” The difficulty with this definition is that it is ambiguous. Mathematically, the length of the “wire” shown in figure 1 is the distance traveled as a point moves along the wire from point A to point B. But the OED definition suggests that the length of the wire might be taken as the straight-line distance from point A to point B. In fact, in some circumstances, this straight-line distance is what should be attended to. For instance, if this wire had to be placed inside a rectangular box, we would probably examine the distance between A and B.

Complicating the situation further is the fact that many words used to describe length are also used to describe time. For instance, we might ask, “Will it take longer to ride the elevator or climb the stairs?” And in many contexts, time and length are interconnected. For example, determining which route from home to school takes longer is a problem concern-
ing time, but the amount of time depends, in part, on the lengths of the routes (as well as speed limits and traffic). Thus, in some contexts, it is difficult for young students to disentangle the concepts of length and time.

Finally, the fact that objects can have various spatial magnitudes associated with them can make constructing meaning for the concept of length difficult for students. For instance, when an adult who understands the concept of length is asked to think about the length of a long narrow box, he or she knows what spatial aspect of the box to focus on. But a child who does not understand the concept of length may have difficulty knowing exactly which spatial magnitude of the box to attend to—the longest linear extent, a horizontal or vertical linear magnitude, the room inside, the area of a face, a diagonal length, or just a vague notion of overall size. Thus, to “see” length requires a conceptualization of length. So the critical question is, How do children develop conceptualizations of length? The answer to this question emerges from careful examination of students’ thinking.

Levels of Sophistication in Students’ Reasoning about Length

There are two fundamentally different types of reasoning about length. Nonmeasurement reasoning does not use numbers. Instead, it involves using visual judgments, direct comparisons, correspondences between parts, and transformations. For instance, students might compare the lengths of objects by placing them next to one another—whichever object appears to extend farther is judged the longest.

Measurement reasoning involves determining the number of unit lengths that fit end to end along an object, with no gaps or overlaps. The process of repeating a unit length end to end along an object is called unit-length iteration. The unit length serves as the unit, or “one,” to be counted. It is assumed that this unit stays constant throughout the measurement process. Measurement reasoning involves not only iterating unit lengths and using measuring instruments but also mentally operating on numerical measurements—for example, adding side lengths to find the perimeter of a rectangle or inferring that opposite sides of a rectangle have equal lengths.

Although students typically start developing nonmeasurement reasoning before measurement reasoning, nonmeasurement reasoning continues to develop even after measurement reasoning appears. Furthermore, the most sophisticated reasoning about length involves the integration of nonmeasurement and measurement reasoning.

I now describe levels of sophistication in students’ development of nonmeasurement and measurement reasoning about length. These levels, outlined in table 1, describe cognitive plateaus reached by students in moving from informal, pre-instructional reasoning to formal mathematical reasoning about length.

**Nonmeasurement Reasoning about Length**

**Nonmeasurement level 0: Holistic visual comparison**

Students’ reasoning about length is appearance based and holistic. It is appearance based because students focus strictly on appearance, or how things look. It is holistic because students focus on whole shapes or objects, not systematically on parts within shapes. Students’ strategies are imprecise and often vague. Frequently, the vagueness is

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<td>N0. Holistic visual comparison</td>
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<td>N2. Comparison by property-based transformations</td>
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Due to the fact that students have not yet separated the length concept from other concepts that occur while dealing with length-related experiences. For instance, students might judge path lengths on the basis of the amount of time or effort they imagine it would take to walk such paths.

As students begin attending to length, they use a variety of visual-holistic strategies. In direct comparison, students compare the lengths of two objects by placing them next to each other. Somewhat more sophisticated is indirect comparison, in which two objects that cannot be compared directly are compared using a third object (such as a string or finger span) to “record” one of the two original lengths and compare it directly to the other original length. Other strategies that students use at this level are comparing the lengths of straight objects by examining only one set of endpoints of the objects; looking only at the endpoints of nonstraight objects and ignoring what occurs between the endpoints; and trying to visualize how long an object, such as a wire, would be if straightened. (See examples in fig. 2; tasks are shown in the appendix on page 146.)

**Nonmeasurement level 1: Comparison by decomposing or recomposing**

**1.1. Rearrangement of path pieces to make new paths that can be directly compared.** Students rearrange—physically, by drawing, or in imagination—some or all path pieces and then directly compare the rearranged paths as wholes. (See examples in fig. 2; tasks are shown in the appendix on page 146.)

**1.2. One-to-one matching of pieces.** Students compare two paths by matching, one by one, pairs of pieces that they think are the same length; they do not transform one path into another. Level 1.2 is more sophisticated than level 1.1 because in level 1.1 students rely on visual manipulation and comparison, whereas in level 1.2 students infer that if the pieces in two paths are the same, then the lengths of the paths are the same. (See fig. 4.)

**Nonmeasurement level 2: Comparison by property-based transformations**

Students compare path lengths by sliding, turning, and flipping shape parts in ways that allow them to infer, on the basis of shape properties, that one transformed shape is congruent to another. (Two figures are congruent if they have exactly the same shape and size.) Students generally do not mention shape properties or the names of transformations (such as “slide,” “flip,” or “turn”), but it is clear that their reasoning is
consistent with such properties and transformations. Note that the slide, turn, and flip transformations are special because they preserve length and congruence—they keep the size and shape of moved objects the same. In particular, when you slide, flip, or turn a 4-cm segment, you get a 4-cm segment.

In figure 5, although Melissa’s reasoning is not completely explicit, carefully examining it shows its sophistication. Melissa’s “move” of segment $p$ to make segment $q$ is actually a slide that not only preserves the length of segment $p$ but also keeps segment $q$ parallel to side $WZ$. Because this slide preserves length, parallelism, and perpendicularity, Melissa can validly infer that segment $q$, together with the vertical segment directly above it, makes segment $XY$, which is the right side of rectangle $WXYZ$. Similarly, she validly infers that sliding $r$ to $s$ makes segment $ZY$, the bottom of rectangle $WXYZ$. Such reasoning usually does not emerge until students have spent a considerable amount of time studying the properties of shapes.

Measurement Reasoning about Length

Measurement level 0: Use of numbers unconnected to unit-length iteration

Students use counting to find lengths; however, their counting does not represent the iteration of a fixed unit length. For instance, students might recite numbers as they continuously move their fingers along a path. Or they might count dots along a path but not as true indicators of unit lengths. (See fig. 6.)

Measurement level 1: Incorrect unit iteration

Students attempt to iterate what they consider to be a unit length along an object or path. However,
because they do not really know what a length unit is or do not properly coordinate iterated units with one another, their iterations contain gaps, overlaps, or units of different lengths. Students often lose track of the size of a unit length while enumerating it.

At this level, students iterate two different types of units. The first type is a shape such as square, rectangle, or cube. Quite often students iterate such a shape without understanding how it corresponds to a unit length. At other times, students may “see” a unit length in a shape, but they have not sufficiently abstracted the unit length to be able to think of it without the shape. The second type of unit that students iterate is a line segment representing a unit length. In this case, students have sufficiently abstracted and isolated length units so that they can explicitly iterate those units, not squares or rectangles or rods. (See fig. 7.)

Measurement level 2: Correct unit iteration

As students iterate unit lengths along an object, they properly coordinate the position of each unit length with the position of the unit that precedes it so that gaps, overlaps, and variations in unit lengths are eliminated. As in the previous level, students iterate two types of units. In the first and more sophisticated type, students have sufficiently abstracted and isolated units so that they can explicitly iterate those units. When iterating units, then, they draw line segments, not squares, rectangles, or rods. In the second and less sophisticated type, students use shapes such as squares, rectangles, or cubes. However, in contrast with what happens in the previous level, at level 2 students clearly match these shapes to unit lengths.

To illustrate the difference between using squares to measure length at levels 1 and 2, consider the unit counting shown in figure 8. On the left, and typical of level 1 reasoning, is an example of counting squares as squares. The squares that are counted do not properly correspond to the unit lengths that make the path. In contrast, on the right, and typical of level 2 reasoning, the counted squares correspond precisely to the unit lengths that make the path (the arrows point to the unit lengths that correspond to the squares). In fact, because students who are reasoning at level 2 focus specifically on length—not, say, on squares as squares—they do not consistently make errors as they count units around corners. (See fig. 9.)
Measurement level 3: Operating on iterations

Students determine some length measurements without explicitly iterating every unit length. They start by iterating and counting length units and then operate on the results of their unit-length iterations numerically or logically.

**Logical operations (making inferences).** On task 6, Marat, a second grader, drew 5 rectangular units along the top of a rectangle, then pointed to the bottom side of the rectangle and said, “10 so far.” The teacher asked Marat why he did not count anything on the bottom. (See fig. 10.) Significantly, Marat created meaning for the bottom side measurement not by iterating units but by inferring what would happen if he iterated units on the basis of properties of a rectangle (that opposite sides have equal lengths). He replaced iteration with inference.

**Numerical operations (adding, subtracting, multiplying, dividing iterated or inferred measurements).** Continuing with this problem, after Marat determined the lengths of all sides of the rectangle—3, 3, 5, and 5—he numerically operated on the numbers by adding them together to find the perimeter. (Fig. 11 shows another student’s work.)

Measurement level 4: Operating on numerical measurements

Students numerically or inferentially operate on length measurements without iterating unit lengths. They make complex, property-based inferences about measurements, often by making transformations or by using properties of geometric shapes. At this highest level of measurement reasoning, students fully integrate and apply the processes from nonmeasurement level 2 with their measurement reasoning. The difference between nonmeasurement level 2 and measurement level 4 is that in measurement level 4 students make inferences about numerical measurements of objects, not the objects themselves. At this level, iteration seems to have fallen in the background. It is as if iteration is considered already completed. (See fig. 12.)

### Conclusion

The levels-of-sophistication framework presented in this article provides a conceptually sound, research-based method to make sense of students’ thinking about length. Once we “locate” where students are in the levels framework—in their constructive itineraries for length—we have a much better idea about how their learning of length—and consequently instruction—should proceed. The levels also help us better understand the difficulties students encounter as they try to make sense of the concept of length and the small steps that many of them must make in achieving mastery of this critical concept. Thus, the levels framework is an important tool for improving instruction and formative assessment as well as effectively diagnosing and remediating students’ difficulties in learning about length.
Appendix

Tasks Relating to Length

**Task 1.** If these were wires and I straightened them, which would be longer, or would they be the same? Each segment between dots is the same.

<table>
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<tr>
<th>Task 2.</th>
<th>Which path from home to school is shortest—the gray path or the dotted path?</th>
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<td><img src="image1" alt="Gray Path" /></td>
<td><img src="image2" alt="Dotted Path" /></td>
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**Task 3.** If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? Why?

**Task 4.** If an ant had to crawl along these paths, which path would be longer for the ant, or would they be the same? Why?

**Task 5.** How many black rods does it take to cover the gray rod?

**Task 6.** How many black rods does it take to cover around the gray rectangle?

**Task 7.** How many black rods does it take to go all the way around the gray rectangle?

**Task 8.** This segment is one centimeter long [point to 1 cm segment]. This rod is 5 centimeters long [show yellow rod]. The line below is 4 rods long. [Show 4 rods on segment, then remove.] How many centimeters long is the line?

**Task 9.** Give the lengths of the sides of a rectangle that has a total distance around of 40 units.

**Task 10.** Find the missing side lengths.

**Task 11.** Which figure has greater perimeter, or do they have the same perimeter?